

TOPOLOGICAL INVARIANCE OF GENERIC NON-UNIFORMLY EXPANDING MULTIMODAL MAPS

STEFANO LUZZATTO AND LANYU WANG

ABSTRACT. We show that a strengthened version of the Collet-Eckmann condition for multimodal maps is topologically invariant. In particular, if f is non-uniformly expanding and the critical points are generic with respect to the absolutely continuous invariant measure, then any map g topologically conjugate to f is also non-uniformly expanding.

1. STATEMENT OF RESULTS

A main theme of this paper is the relation between the *topological* structure of a dynamical system and its *measure-theoretic* and *ergodic* properties. Throughout the paper we shall be concerned with the class \mathcal{S} of C^3 interval maps $f : I \rightarrow I$ with negative Schwarzian derivative and a finite number of non-flat critical points with possibly different orders, we give the precise definitions in section 1.9 on page 6 below. Our main result is to show that a sub-class of maps in \mathcal{S} admit equivalent characterizations in terms of some topological and metric properties of the critical orbits, which in turn imply several ergodic properties such as the existence of an absolutely continuous invariant measure. This subclass is quite abundant in \mathcal{S} in the sense that general one-parameter families of maps will belong to it for a positive Lebesgue measure set of parameters.

1.1. Non-uniformly expanding maps. A *physical measure* for f is any ergodic invariant measure μ with a basin $B(\mu)$ of positive Lebesgue measure, where $B(\mu) = \{x : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{f^i(x)} = \mu\}$, and the limit is in the weak- \star topology. Maps in \mathcal{S} can admit various kinds of physical measures such as Dirac- δ measures supported on attracting fixed points or absolutely continuous measures. There also exist examples for which f admits no physical measures at all [HofKel90b, Kel00]. It was shown in [BloLyu89] that the number of physical measures for maps in \mathcal{S} is finite and bounded by the number of critical points.

Definition 1. We say that $f : I \rightarrow I$ is *(non-uniformly) expanding* if there exists a finite number μ_1, \dots, μ_q , $q \geq 1$, of ergodic absolutely continuous

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invariant probability measures such that the union of their basins has full Lebesgue measure and such that

$$(1) \quad 0 < \int \log |Df| d\mu < \infty$$

for each $\mu = \mu_i$, $i = 1, \dots, q$.

For simplicity we assume that $q = 1$ and so there is only one absolutely continuous invariant probability measure whose basin has full measure and which satisfies (1). In the general case our results hold for the restriction of f to the basins $B(\mu_i)$ of each ergodic measure μ_i . In fact our results hold for the restriction of f to the basin of any ergodic absolutely continuous invariant probability measure satisfying (1) with no additional assumptions on the dynamics outside this basin.

1.2. Topological invariance of non-uniform expansivity. We recall that $f : I \rightarrow I$ and $g : J \rightarrow J$ are *topologically conjugate*, or $f \sim g$, if there exists a homeomorphism $h : I \rightarrow J$ such that $h \circ f = g \circ h$. A topological conjugacy preserves topological properties of maps such as dense and periodic orbits but in principle there is no reason for any measure-theoretic properties such as the existence of absolutely continuous invariant measures to be preserved. We want to address the following question:

Suppose that f is non-uniformly expanding and that $f \sim g$. Does this imply that g is non-uniformly expanding?

A counterexample of Bruin [Bru98a] shows that the answer is negative in general even in the unimodal case. However we prove the following

Theorem 1. *Suppose that f is non-uniformly expanding, $f \sim g$, and that all critical points of f are generic with respect to μ in the sense that*

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log |Df_{f^i(c)}| = \int \log |Df| d\mu.$$

Then g is non-uniformly expanding.

Notice that (2) holds for μ almost all points and in fact, by the absolute continuity of μ and the assumption that $B(\mu)$ has full Lebesgue measure, for Lebesgue almost all points. This justifies thinking of it as a “generic” condition.

We remark that the proof does not imply that the conjugacy h is absolutely continuous. Rather, it is based on showing that g satisfies certain conditions which can be shown by independent arguments to imply that g is non-uniformly expanding.

1.3. The unimodal case. The only existing positive results on the question of the topological invariance of non-uniform expansivity are in the unimodal setting. The first example of a (topological class of a) map with an absolutely continuous invariant measure which is preserved by topological

conjugacy was constructed by Bruin [Bru94], another such class is defined in [Bru98b]. Sands [San95] formulated a (topological) condition on the kneading sequence of the map, i.e. the symbolic itinerary of the critical point, and showed that this implied the following condition

$$(3) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log |Df_{f^i(c)}| > \lambda > 0$$

for the (single) critical point c (and this implies non-uniform expansivity, see the next section). Notice that (3) is strictly weaker than (2). It follows from our results that Sands' topological condition is actually equivalent to a metric condition which we will call Tsujii's condition, to be defined below, and which is strictly stronger than (3) and strictly weaker than (2). It was then shown in [NowPrz98] that (3) itself is topologically invariant, by bringing together results of previous papers [NowSan98, PrzRoh98] and observing that these results formed a closed chain of implications which included a condition which is clearly invariant by topological conjugacy.

These result do not however generalize to the multimodal setting: a counterexample was recently constructed in [PrzRivSmi03] exhibiting a pair of quasi-symmetrically (in particular topologically) conjugate multimodal maps of which one satisfies (3) and the other does not. We shall show nevertheless that the slightly stronger Tsujii condition mentioned above is topologically invariant for completely general multimodal maps. This condition is satisfied by generic maps (2). First we give a more in-depth discussion of condition (3) and its relation to (2).

1.4. Collet-Eckmann conditions. Condition (3) is more often formulated as

$$(4) \quad |Df_{f(c)}^n| \geq C e^{\lambda n} \quad \text{for } C, \lambda > 0, \quad \forall \quad n \geq 1.$$

It is traditionally known as the Collet-Eckmann condition since it was first introduced in [ColEck83] and shown, together with some additional conditions which were later shown to be unnecessary [NowStr88], to imply non-uniform expansivity for unimodal maps. The generalization of these results to the multimodal setting has been somewhat problematic for various technical reasons and only recently it has been shown that condition (4) assumed to hold for *every* critical point does indeed imply non-uniform expansivity, see [BruLuzStr03] for the case when all critical points have the same order and [BruStr01] for the general case.

We remark that in the general multimodal case condition (4) is always satisfied by Lebesgue almost every point if f is non-uniformly expanding since it is equivalent to (3) which follows from (2) which holds for Lebesgue almost every point by (1) and the assumptions on μ . This justifies the use of the term *expanding* for such a map f . The *non-uniformity* follows from the fact that the convergence in (3) is not necessarily uniform and thus in general we have that the constant C in (4) is actually a function C_x which

depends on x and such that $\inf_x C_x = 0$: it might be necessary to wait for an arbitrary long time before the exponential behaviour becomes apparent.

The difference between (2) and (3) is quite subtle and does not usually play an important role in many argument which take advantage of the exponential growth property (4). In our results however, this distinction will play a very important role. We shall define below a set of conditions which together fall strictly between (2) and (3).

1.5. The Main Theorem. The statement in Theorem 1 is somewhat unsatisfactory due to the asymmetry between the assumptions and the conclusions: it is not the case in general that the critical points for g are generic with respect to the corresponding absolutely continuous invariant measure for g . Moreover it does not include some simple, albeit exceptional, cases, such as when the critical points are non-recurrent, which are clearly topologically invariant conditions and are known to imply non-uniform expansivity. We shall therefore isolate the two main properties of genericity which are actually used in the proof and show that these properties are indeed topologically invariant. One of these properties is the Collet-Eckmann condition (3) (or (4)). The other is a condition on the recurrence to the critical set.

Let $d(x, y) = |y - x|$ denote the standard Euclidean distance in I , let \mathcal{C} denote the set of critical points of f and let $d(x) = d(x, \mathcal{C}) = \min\{d(x, c), c \in \mathcal{C}\}$. For any $\delta > 0$, let $\mathcal{C}_\delta = \{x : d(x) \leq \delta\}$ denote a *metric neighbourhood* of the critical set. We say that a (critical) point c satisfies the *slow recurrence* condition if

$$(5) \quad \lim_{\delta \rightarrow 0^+} \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{\substack{1 \leq i \leq n \\ f^i(c) \in \mathcal{C}_\delta}} \log d(f^i(c)) = 0.$$

Almost every point satisfies (5) since $\log |Df_{f^i(x)}|$ and $\log d(f^i(x))$ are uniformly comparable by the non-flatness of the critical points (7), Birkhoff's ergodic Theorem implies that

$$\frac{1}{n} \sum_{f^i(x) \in \mathcal{C}_\delta} \log |Df_{f^i(x)}| \rightarrow \int_{\mathcal{C}_\delta} \log |Df| d\mu$$

and the integrability condition (1) implies that $\int_{\mathcal{C}_\delta} \log |Df| d\mu \rightarrow 0$ as $\delta \rightarrow 0$. It is therefore in some sense also a generic condition.

We say that a map satisfies condition CE if all its critical points satisfy the Collet-Eckmann condition (3) and SR if all its critical points satisfy the Slow recurrence conditions (5). We are now ready to state our main result.

Main Theorem. *Suppose that f satisfies both CE and SR. Then every map g topologically conjugate to f also satisfies both CE and SR.*

Thus the simultaneous occurrence of both conditions CE and SR is topologically invariant. We stress the fact that neither CE nor SR on their own are topologically invariant.

1.6. Tsujii's Conditions. The measure-theoretical abundance of multimodal maps satisfying conditions CE and SR simultaneously was proved by Tsujii [Tsu93] for quite general families of multimodal maps, building on a theme of proving the abundance of maps satisfying various kinds of expansivity and recurrence conditions, all aimed at ultimately implying some form of non-uniform expansivity [Jak81, BenCar85, ThiTreYou94, Tsu93, Luz00]. We mention also that non-uniformly expanding maps are in general very structurally unstable in the sense that arbitrarily small perturbations which do not preserve the topological structure of the map can destroy the expansivity, see [GraSwi97, Lyu02, Koz03, She, KozSheStr03].

We conjecture that Tsujii's conditions might play a similar role in the multimodal setting to that played by the Collet-Eckmann conditions in the unimodal case. Several recent counterexamples show that some key properties of unimodal CE maps do not extend to the multimodal setting; perhaps these counterexamples may be overcome by assuming Tsujii's conditions instead. Our main theorem is precisely an example in which this is the case. Our arguments also indicate possible strategies for proving other technical properties for maps satisfying Tsujii's conditions, such as *backward Collet-Eckmann* which is equivalent to CE in the unimodal case [NowSan98] but does *not* follow from straightforward CE in the multimodal case [BruStr01], *bounded backward contraction* and *exponential decay of correlations* proved in [BruStr03] and [BruLuzStr03] respectively for CE multimodal maps under the additional assumption that all critical points have the same order. Counterexamples exist showing that bounded backward contraction may fail for CE maps with critical points of different order [BruStr03]. These properties will be investigated in future papers.

1.7. Sands' condition. Our approach is quite distinct from the proof in [NowPrz98] of the invariance of the CE condition in the unimodal case. The present strategy is more direct and in the spirit of Sands' thesis [San95] in which, in the unimodal setting, a topological condition is formulated in terms of the kneading sequence of the critical point and this condition is shown to imply the Collet-Eckmann condition. The relation between Sands' topological condition and our Slow Recurrence condition (5) was studied in [Wan01] in the unimodal case, and a generalization of the topological condition given in the bimodal case in [Wan96]. Here we push this whole approach significantly further by giving a more general definition of this condition which in particular applies to the multimodal setting, we call it a Topological Slow Recurrence (TSR) condition, see (14) on page 11. By construction this condition is topologically invariant and thus our main result follows from proving the double implication

Main Technical Theorem. *For any $f \in \mathcal{S}$ we have*

$$(6) \quad CE + SR \iff TSR.$$

1.8. Overview of the paper. In section 2 we give all the details of the combinatorial structure and define condition TSR. We give a brief summary of the key ideas here. The critical set \mathcal{C} defines a partition \mathcal{P} of I which allows us to define a symbolic itinerary for all points which are not preimages of \mathcal{C} and to define cylinder sets made up of points which share the same itinerary up to some finite time. The whole structure of cylinder sets is topological. There is then a natural *topological* separation time: any point x near \mathcal{C} shadows the orbit of the critical point for a time defined by the number of iterates for which x and c have the same symbolic itinerary. This shadowing time tends to ∞ as x tends to c . In particular the symbolic sequence associated to any given point contains information about its pattern of recurrence to the critical set: if it contains long finite blocks which coincide with long initial blocks of the symbolic sequences of one of the critical point then it must have come quite close to the critical set. This simple idea is used in a crucial way in the definition of the condition TSR.

In the remaining three sections we proceed to prove the three steps required to show (6). We first show that CE + SR implies TSR, then that TSR implies SR and finally that TSR implies CE. A particular effort has been made here to make the presentation as self-contained as possible and as much as possible independent of specialized technical notation and constructions familiar mainly to established specialists in one-dimensional dynamics.

1.9. S-multimodal maps with non-flat critical points. Before ending this section we give the formal definition of the class \mathcal{S} of maps under consideration and fix some fairly standard notation and definitions. We shall consider *S-multimodal maps with non-flat critical points* by which we mean maps f satisfying the following conditions. Let $f : I \rightarrow I$ be a C^3 map, where $I \subset \mathbb{R}$ is a closed interval. We shall assume without loss of generality that $I = [0, 1]$. We call $c \in I$ a critical point of f if $Df(c) = 0$. Denote by \mathcal{C} the set of all critical points. f has finite number (say q) of *non-flat critical points* $0 < c_1 < c_2 < \dots < c_q < 1$ if there exist $L_i > 1$, $l_i > 1$ and a neighbourhood $V(c_i)$ of c_i in which

$$(7) \quad |x - c_i|^{l_i-1}/L_i \leq |Df(x)| \leq L_i|x - c_i|^{l_i-1}$$

for any $x \in V(c_i)$ and $1 \leq i \leq q$. f has *negative Schwarzian derivative*, i.e. $S(f)(x) = (D^3f(x)/Df(x)) - (3D^2f(x)/2Df(x))^2 < 0$ for $x \in (0, 1) \setminus \mathcal{C}$. We remark that this condition implies in particular the non-existence of any inflection points. Thus all critical points are turning points, i.e. points at which the maps fails to be a local homeomorphism.

As part of the argument we shall need to obtain some derivative estimates about the iterates of f along the forward orbit of the critical values. If these orbits map to a periodic point then the estimates are easily obtained. Otherwise we need to consider a two-sided neighbourhood of each critical value and the forward images of this neighbourhood. We assume therefore that the forward orbits of all the critical points are disjoint from the boundary of

I or, if not, that the boundaries of I are fixed points. This is not restrictive since, if it is not the case, we can extend f to a map \hat{f} defined on neighbourhood \hat{I} of I such that the forward orbits of all the critical points are disjoint from the boundary of \hat{I} and the dynamics on I is unaffected.

2. COMBINATORIAL STRUCTURE

In this section we describe in detail the combinatorial structure associated to any S-multimodal map with non-flat critical points and having no neutral or attracting periodic orbits. We emphasize that everything we describe here relies on no other assumptions. Some of the definitions are closely related to the so-called *cutting/co-cutting times* used in [Hof80, HofKel90b, Bru95a, San95, NowSan98] for unimodal maps; we give a generalization to the multimodal setting and, we believe, a simplification of the way those concepts are introduced and applied.

We fix for the rest of the paper a constant

$$(8) \quad \delta_0 > 0$$

sufficiently small so that the critical neighbourhood \mathcal{C}_{δ_0} has precisely q disjoint connected components; in particular any $x \in \mathcal{C}_\delta$ for $\delta < \delta_0$ is associated unambiguously to one particular critical point. For any x we shall use the notation $x^i = f^i(x)$, especially, $c^i = f^i(c)$ for a critical point c .

2.1. Symbolic dynamics. The critical points define a natural partition

$$\mathcal{I} = \{I_0, \dots, I_q\}$$

of I into $q + 1$ subintervals. This allows us to associate a symbolic *itinerary* to any point not in the preimage of a critical point by letting

$$\underline{a}(x) = a_0 a_1 a_2 \dots \text{ where } a_i = k \text{ if } f^i(x) \in I_k \text{ for all } i \geq 0.$$

For convenience we associate to each critical point two sequences which differ only in the first term, so that we think of the partition elements as closed intervals and think of each critical point as belonging to both of the adjacent intervals. We define the symbolic *separation time* between two points as

$$s(x, y) = \min\{k \geq 0 : a_k(x) \neq a_k(y)\}$$

as long as the itineraries $\underline{a}(x)$ and $\underline{a}(y)$ are both defined. If $y = c$ is a critical point and x is in one of the adjacent partition elements then we naturally consider the symbolic itinerary for c for which $s(x, y) = s(x, c) \geq 1$. In particular we define the time it takes for a point x to “separate” from the critical set \mathcal{C} by

$$s(x) = s(x, \mathcal{C}) = \max\{s(x, c) : c \in \mathcal{C}\}.$$

Notice that $s(x) \rightarrow \infty$ as $d(x) \rightarrow 0$. We define *topological neighbourhoods* \mathcal{C}_n of the critical set \mathcal{C} by

$$\mathcal{C}_n = \{x : s(x) \geq n\}.$$

We fix a constant N_0 sufficiently large so that

$$(9) \quad \mathcal{C}_{N_0} \subset \mathcal{C}_{\delta_0},$$

recall (8), and such that every point x with $s(x) \geq N_0$ is associated to some particular critical point c and $d(x) = d(x, c) \leq \delta_0$ and $s(x) = s(x, c)$.

For any given finite sequence $a_0 \dots a_n$ with $a_i \in \{0, \dots, q\}$ we define the *cylinder set*

$$\hat{I}_{a_0 \dots a_n}^{(n)} = \{x : f^i(x) \in I_{a_i}, 0 \leq i \leq n\}$$

Notice that such a cylinder set may be empty. For a given point x with a symbolic itinerary $\underline{a}(x)$ we define the n 'th order cylinder set of x as

$$\hat{I}^{(n)}(x) = \hat{I}_{a_0 \dots a_n}^{(n)}.$$

This is always a two sided neighbourhood of x . We shall often write

$$\hat{I}_-^{(n)}(x) \quad \text{and} \quad \hat{I}_+^{(n)}(x)$$

to denote the part of $\hat{I}^{(n)}(x)$ to the left and right respectively of the point x . Clearly $\hat{I}^{(n+1)}(x) \subseteq \hat{I}^{(n)}(x)$ for any n . Moreover, by the non-existence of wandering intervals [Guc79, MelStr93] the preimages of the critical set are dense and therefore we have

$$|\hat{I}^{(n)}(x)| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad \forall x \in I.$$

In particular the cylinder sets $\hat{I}^{(n)}(x)$ define a nested sequence of neighbourhoods of the point x .

2.2. Partitions. We now fix some arbitrary point x not contained in any preimage of the critical set. We fix some notation related to the partition $\mathcal{P}(x)$ of a neighbourhood of x defined naturally by the cylinder sets. More precisely, consider a generic cylinder set $\hat{I}^{(n-1)} = \hat{I}^{(n-1)}(x)$. By definition the images $f^j(\hat{I}^{(n-1)})$ all belong to the same element of \mathcal{I} for all $j \leq n-1$. Then there are two possibilities:

- (1) $f^n(\hat{I}^{(n-1)})$ does not contain any critical points in its interior. In this case all points continue to share the same combinatorics and we have $\hat{I}^{(n)} = \hat{I}^{(n-1)}$, and thus in particular

$$\hat{I}^{(n-1)} \setminus \hat{I}^{(n)} = \emptyset.$$

- (2) $f^n(\hat{I}^{(n-1)})$ contains one or more critical points in its interior. In this case the cylinder set $\hat{I}^{(n)} \subset \hat{I}^{(n-1)}$ is given precisely by those points of $\hat{I}^{(n-1)}$ which fall into the same element of \mathcal{I} as $f^n(x)$ and

$$\hat{I}^{(n-1)} \setminus \hat{I}^{(n)} \neq \emptyset.$$

In this case $\hat{I}^{(n-1)} \setminus \hat{I}^{(n)}$ may have either one or two connected components.

We index the times at which case 2 above occurs by two sequences

$$\mathcal{N}_x^- = \{n_i^-\}_{i=1}^\infty \quad \text{and} \quad \mathcal{N}_x^+ = \{n_i^+\}_{i=1}^\infty$$

where

$$n_i^- \text{ denotes a time for which } \hat{I}_-^{(n_i^- - 1)} \setminus \hat{I}_-^{(n_i^-)} \neq \emptyset$$

and

$$n_i^+ \text{ denotes a time for which } \hat{I}_+^{(n_i^+ - 1)} \setminus \hat{I}_+^{(n_i^+)} \neq \emptyset.$$

For a general point x , the two sequences \mathcal{N}_x^- and \mathcal{N}_x^+ are independent even though there may well be one or more pairs i, j such that $n_i^- = n_j^+$. For the special case of the partition $\mathcal{P}(c)$ associated to one of the critical points c , we actually have $\mathcal{N} = \mathcal{N}_c^- = \mathcal{N}_c^+$. For each such time n_i^\pm we define the intervals

$$I^{(n_i^\pm)} = \hat{I}_\pm^{(n_i^\pm - 1)} \setminus \hat{I}_\pm^{(n_i^\pm)}$$

which define precisely the elements of the partition

$$\mathcal{P}(x) = \left\{ I^{(n_i^\pm)} : n_i^\pm \in \mathcal{N}_x^\pm \right\}$$

of a neighbourhood of x , refer to Figure 1.

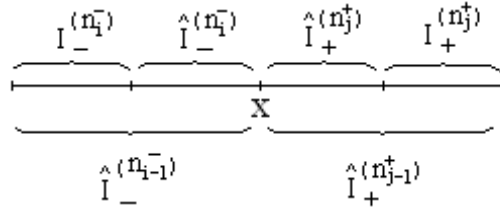


FIGURE 1. Cylinder sets and partition elements around a point x .

2.3. Shadowing times. We call the elements of the sequences \mathcal{N}_x^- and \mathcal{N}_x^+ respectively the *left and right shadowing times* associated to x because every point in $I^{(n_i^\pm)}$ “shadows” the point x for n_i^\pm iterations:

$$(10) \quad s(y, x) = n_i^\pm \iff y \in I^{(n_i^\pm)}(x)$$

The partition $\mathcal{P}(x)$ (and in particular $\mathcal{P}(c)$ for some critical point c) can be thought of as topological versions of the partitions defined by the *binding periods* of Benedicks and Carleson [BenCar85]. Here we lose some good properties of binding periods such as uniformly bounded distortion but gain in other ways, as shall become clear below. First we state three relatively straightforward but not immediately obvious properties of this partition which will play an important role. First of all notice that

$$(11) \quad f^{n_{i+1}^\pm} \text{ is monotone on } \hat{I}_\pm^{(n_i^\pm)}.$$

Indeed, by construction the boundary points of each partition element $I^{(n_i^\pm)}$ are preimages of some critical point, and one of the boundary points of each cylinder set $\hat{I}_\pm^{(n_i^\pm)}$ is a preimage of some critical point of order n_i^\pm . Moreover, there are no points in the interior of $\hat{I}_\pm^{(n_i^\pm)}$ which map to a critical point before time n_{i+1}^\pm . Thus (11) follows.

Secondly, for any y close to x we have

$$(12) \quad |\hat{I}_\pm^{s(y,x)-1}| \geq d(y, x) \geq |\hat{I}_\pm^{s(y,x)}|.$$

Indeed, by construction we have $\hat{I}_\pm^{(m)} = \hat{I}_\pm^{(n_{i-1}^\pm)}$ for $n_{i-1}^\pm \leq m < n_i^\pm$. and so in particular $\hat{I}_\pm^{(n_{i-1}^\pm)} = \hat{I}_\pm^{(n_i^\pm-1)}$. Therefore, for any $y \in I^{(n_i^\pm)}$ the relation (10) gives

$$y \in I^{(n_i^\pm)} = \hat{I}_\pm^{(n_{i-1}^\pm)} \setminus \hat{I}_\pm^{(n_i^\pm)} = \hat{I}_\pm^{(n_i^\pm-1)} \setminus \hat{I}_\pm^{(n_i^\pm)} = \hat{I}_\pm^{s(y,x)-1} \setminus \hat{I}_\pm^{s(y,x)}$$

which implies (12).

Thirdly, for every $i \geq 1$ such that $n_{i+1}^\pm - n_i^\pm \geq N_0$ we have

$$(13) \quad s(x^{n_i^\pm}) = n_{i+1}^\pm - n_i^\pm.$$

To see this consider the two intervals $[y, x] := \hat{I}_\pm^{(n_{i+1}^\pm)} \subset \hat{I}_\pm^{(n_i^\pm)} =: [z, x]$. By construction we have $f^{n_i^\pm}(z) = \tilde{c}$ for some critical point $\tilde{c} \in \mathcal{C}$ and thus $f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)}) = f^{n_i^\pm}[z, x] = [\tilde{c}, x^{n_i^\pm}]$ and $f^{n_i^\pm}(y) \in (\tilde{c}, x^{n_i^\pm})$, i.e. the interval $[\tilde{c}, x^{n_i^\pm}]$ contains the point $f^{n_i^\pm}(y)$ in its interior. By (11), $f^{n_{i+1}^\pm}$ is monotone on $\hat{I}_\pm^{(n_i^\pm)}$ which implies in particular that $f^j(\hat{I}_\pm^{(n_i^\pm)})$ cannot contain any critical point in its interior before time n_{i+1}^\pm , i.e. $f^j(\hat{I}_\pm^{(n_i^\pm)}) \cap \mathcal{C} = \emptyset$ for all $j < n_{i+1}^\pm$. At this time we have $f^{n_{i+1}^\pm}(y) = f^{n_{i+1}^\pm - n_i^\pm}(f^{n_i^\pm}(y)) = \hat{c}$ for some (other) critical point $\hat{c} \in \mathcal{C}$. Therefore $n_{i+1}^\pm - n_i^\pm$ is exactly the first time at which the iterates of \tilde{c} and of $x^{n_i^\pm}$ fall on different sides of some critical point, and therefore is exactly the separation time: $s(x^{n_i^\pm}, \tilde{c}) = n_{i+1}^\pm - n_i^\pm$. If $n_{i+1}^\pm - n_i^\pm \geq N_0$ this implies that $s(x^{n_i^\pm}) = s(x^{n_i^\pm}, \tilde{c})$ as in (13). See Figure 2.

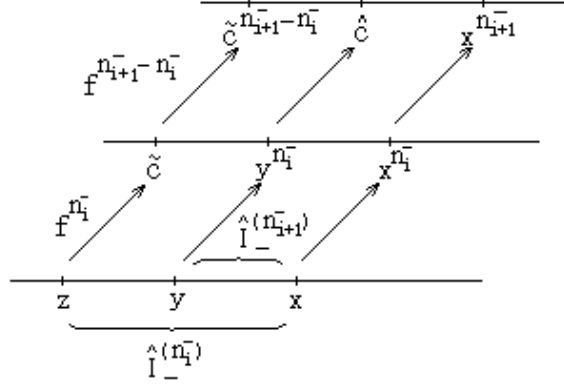


FIGURE 2. Iteration of the left part of corresponding cylinder set for some left shadowing time at x .

2.4. Topological Slow Recurrence. We are now ready to formulate the *Topological Slow Recurrence* condition for an arbitrary point x :

$$(14) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{\substack{1 \leq j \leq n \\ x^j \in \mathcal{C}_m}} s(x^j) = 0.$$

Definition 2. We say that the map f satisfies condition TSR if all the critical points satisfy (14)

Notice that this depends only on the orbit of the critical points with respect to the partitions $\mathcal{P}(c)$ for $c \in \mathcal{C}$. It is therefore invariant under topological conjugacy.

3. CE + SR IMPLIES TSR

Lemma 1. *Let f satisfy condition CE. Then there exists a constant $\bar{\kappa} > 0$ such that*

$$(15) \quad s(c^j) \leq \bar{\kappa} \log d(c^j)^{-1} \quad \forall c \in \mathcal{C}, \forall j \geq 1.$$

We shall use the following

Sublemma 1.1. *[NowPrz98] There exists constants $C, \bar{\xi} > 0$ such that for all intervals J and integers $s \geq 1$ such that $f^s|_J$ is monotone we have $|J| \leq C e^{-s\bar{\xi}}$.*

Proof of Lemma 1. We only need to prove the result for $s(c^j) \geq N_0$ as the choice of the constant $\bar{\kappa}$ can take into account smaller values. Then there is a well-defined critical point \tilde{c} “closest” to c^j i.e. $s(c^j) = s(c^j, \tilde{c})$, and $d(c^j) = d(c^j, \tilde{c}) = |c^j - \tilde{c}|$. The map $f^{s(c^j)}$ is monotone on the interval $[c^j, \tilde{c}]$ and therefore Sublemma 1.1 implies $d(c^j) = |c^j - \tilde{c}| \leq Ce^{-s\xi}$. \square

Corollary 1.1. *Let f satisfy conditions CE and SR. Then it satisfies condition TSR.*

Proof. By condition SR, for every $\varepsilon > 0$ there exist $n_\varepsilon, \delta_\varepsilon > 0$ such that

$$\frac{1}{n} \sum_{\substack{1 \leq j \leq n \\ f^j(c) \in \mathcal{C}_{\delta_\varepsilon}}} \log |d(f^j(c))|^{-1} < \varepsilon \quad \forall c \in \mathcal{C}, \quad \forall n \geq n_\varepsilon.$$

Therefore, choosing T_ε sufficiently large so that $\mathcal{C}_{T_\varepsilon} \subseteq \mathcal{C}_{\delta_\varepsilon}$ and using (15) we have

$$\frac{1}{n} \sum_{\substack{1 \leq j \leq n \\ f^j(c) \in \mathcal{C}_{T_\varepsilon}}} s(c^j) \leq \frac{\bar{\kappa}}{n} \sum_{\substack{1 \leq j \leq n \\ f^j(c) \in \mathcal{C}_\delta}} \log |d(c^j)|^{-1} < \bar{\kappa}\varepsilon.$$

Since ε is arbitrary, this implies TSR. \square

4. TSR IMPLIES SR

Lemma 2. *Let f satisfy condition TSR. Then there exists a constant $\underline{\kappa} > 0$ such that*

$$(16) \quad s(c^j) \geq \underline{\kappa} \log d(c^j)^{-1} \quad \forall c \in \mathcal{C}, \quad \forall j \geq 1.$$

Arguing as in the proof of Corollary 1.1 we then get

Corollary 2.1. *Let f satisfy condition TSR. Then it satisfies condition SR.*

We first give two preliminary results which will be used in the proof.

Sublemma 2.1. *There exists constants $C, \xi > 0$ such that for any interval J and any positive integer n such that J is a one-side neighbourhood of a critical point c , $f^n(J)$ contains a one-sided neighbourhood of the same critical point c and $f^{n+1}(J) \supset f(J)$, we have $|J| \geq Ce^{-n\xi}$ (we do not assume that $f^{n+1}|_J$ is monotone).*

Proof. By the non flatness of the critical points, there exists a constant $\theta > 0$ depending only on f such that in either case we have $\theta|J| \leq |f^n(J)|$. Moreover, letting $D = \max_{x \in I} |Df(x)|$, the mean value theorem implies $|f^n(J)| \leq D^{n-1}|f(J)|$, and, using the non-flatness of the critical points again, we have $|f(J)| \leq L|J|^\ell$ for some constants ℓ, L depending only on f . Combining these estimates give $\theta|J| \leq LD^{n-1}|J|^\ell$ and therefore $|J|^{\ell-1} \geq \theta LD^{-n+1}$ or $|J| \geq (\theta L^{-1}D)^{1/(\ell-1)} D^{-n/(\ell-1)}$. \square

Sublemma 2.2. *Let f satisfy condition TSR. Then for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that*

$$(17) \quad n_{i+1}(c) - n_i(c) \leq \varepsilon n_i(c) \quad \forall c \in \mathcal{C}, \quad \forall n_i \geq N_\varepsilon.$$

Proof. From TSR we have $s(c^j)/j \rightarrow 0$ as $j \rightarrow \infty$. By (13) this immediately implies the statement. \square

Proof of Lemma 2. We first define in an iterative process which may stop according to one of two possible stopping rules. We then show that in both cases the conclusions of the Lemma are satisfied. We fix some $\varepsilon \in (0, 1)$ and let N_ε be the corresponding integer from Sublemma 2.2.

Step 1. Let $s_0 = s(c^j)$.

- If $s_0 < \max\{N_0, N_\varepsilon\}$ go to step 3.

Otherwise we argue as follows. Since $s_0 \geq N_0$, there is a well defined critical point $c_{(0)}$ such that $d(c^j) = d(c^j, c_{(0)})$ and $s(c^j) = s(c^j, c_{(0)})$. By (10) there exists some $n_{i_0} \in \mathcal{N}_{c_{(0)}}$ with $n_{i_0} = s_0$ and

$$\hat{I}^{n_{i_0}} \subset (c^j, c_{(0)}) \subset \hat{I}^{n_{i_0}-1}.$$

Here we omit the subscripts \pm not to overload the notation, and let $\hat{I}^{n_{i_0}} = \hat{I}_+^{n_{i_0}}$ or $\hat{I}_-^{n_{i_0}}$ according to the relative positions of c^j and $c_{(0)}$. By construction

$$f^{s_0}(\hat{I}^{n_{i_0}}) = (c_{(0)}^{s_0}, c_{(1)})$$

for some critical point $c_{(1)}$.

- If $c_{(1)} = c_{(0)}$ go to step 2.

Otherwise we repeat the algorithm with $c_{(0)}^{s_0}$ playing exactly the role of c^j above. More precisely, we consider the separation time $s_1 = s(c_{(0)}^{s_0})$. Since $s_0 > N_\varepsilon$, (13) and (17) give

$$s_1 = n_{i_0+1} - n_{i_0} \leq \varepsilon n_{i_0} = \varepsilon s_0.$$

- If $s_1 < \max\{N_0, N_\varepsilon\}$ go to step 3.

Otherwise there exists some $n_{i_1} \in \mathcal{N}_{c_{(1)}}$ such that $n_{i_1} = s_1$ and

$$\hat{I}^{n_{i_1}} \subset (c_{(0)}^{s_0}, c_{(1)}) \subset \hat{I}^{n_{i_1}-1}.$$

Again, by construction

$$f^{s_1}(\hat{I}^{n_{i_1}}) = (c_{(1)}^{s_1}, c_{(2)})$$

for some critical point $c_{(2)}$.

- If $c_{(2)}$ equals either $c_{(0)}$ or $c_{(1)}$ go to step 2.

Otherwise we repeat the process again with $c_{(1)}^{s_1}$ playing the role of $c_{(0)}^{s_0}$ and continue in this way until we go to either step 2 or step 3. In the general case we have $s_j = s(c_{(j-1)}^{s_{j-1}})$ and

$$(18) \quad s_j = n_{i_{j-1}+1} - n_{i_{j-1}} \leq \varepsilon n_{i_{j-1}} \leq \varepsilon^j s_0.$$

- If $s_j < \max\{N_0, N_\varepsilon\}$ go to step 3.

Otherwise there exists some $n_{i_j} \in \mathcal{N}_{c_{(j)}}$ with $n_{i_j} = s_j$ such that

$$\hat{I}^{n_{i_j}} \subset (c_{(j-1)}^{s_{j-1}}, c_{(j)}) \subset \hat{I}^{n_{i_{j-1}}} \text{ and } f^{s_j}(\hat{I}^{n_{i_j}}) = (c_{(j)}^{s_j}, c_{(j+1)})$$

for some critical point $c_{(j+1)}$.

• If $c_{(j+1)}$ equals any one of $c_{(0)}, c_{(1)}, \dots, c_{(j)}$ go to step 2.

The process has to stop in a maximum of $q + 1$ steps as by that time the above condition is necessarily satisfied. See Figure 3.

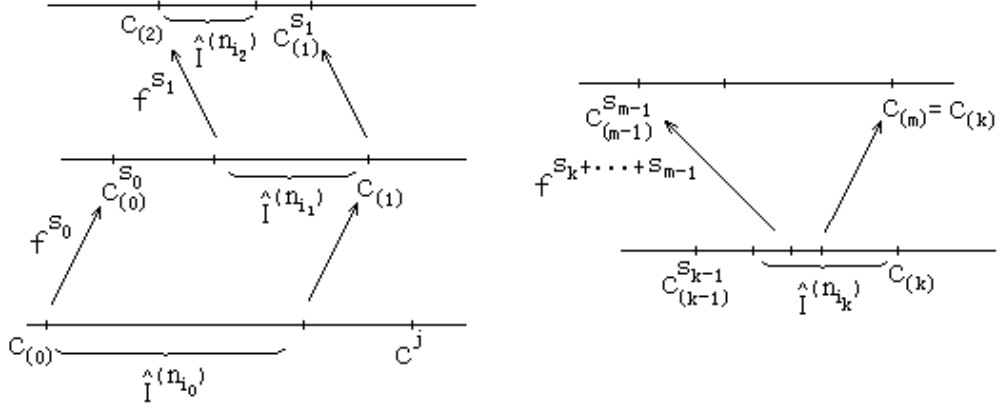


FIGURE 3. Iterative process at the critical point, (a) step 1, (b) step 2.

Step 2. We suppose here that the procedure described above gives constants $0 \leq k < m \leq q + 1$ such that $c_{(m)} = c_{(k)}$ and therefore we have one-sided neighbourhoods $(c_{(k-1)}^{s_{k-1}}, c_{(k)})$ and $(c_{(m-1)}^{s_{m-1}}, c_{(m)}) = (c_{(m-1)}^{s_{m-1}}, c_{(k)})$ of the same critical point $c_{(k)}$. Moreover, by construction we have that $f^{s_k + \dots + s_{m-1}}(c_{(k-1)}^{s_{k-1}}, c_{(k)}) \supset (c_{(m-1)}^{s_{m-1}}, c_{(m)}) = (c_{(m-1)}^{s_{m-1}}, c_{(k)})$ and therefore we are in a position to apply the estimates of Sublemma 2.1 to the interval $J = (c_{(k-1)}^{s_{k-1}}, c_{(k)})$ if we can show that

$$(19) \quad f(c_{(m-1)}^{s_{m-1}}, c_{(k)}) \supset f(c_{(k-1)}^{s_{k-1}}, c_{(k)}).$$

To see that this is the case we recall that s_j satisfies (18) for all $0 \leq j < m$ (for otherwise we would have gone straight to Step 3) and thus in particular the separation time $s(c_{(m-1)}^{s_{m-1}}, c_{(k)}) = s(c_{(m-1)}^{s_{m-1}}, c_{(m)}) = s_m$ is strictly less than $s(c_{(k-1)}^{s_{k-1}}, c_{(k)}) = s_k$. This implies that the separation time $s(f(c_{(m-1)}^{s_{m-1}}), f(c_{(k)}))$ is strictly less than $s(f(c_{(k-1)}^{s_{k-1}}), f(c_{(k)}))$ where

both $(f(c_{(m-1)}^{s_{m-1}}), f(c_{(k)}))$ and $(f(c_{(k-1)}^{s_{k-1}}), f(c_{(k)}))$ are one-sided neighbourhood (on the same side) of the critical value $f(c_{(k)})$. Clearly this implies that $|f(c_{(k-1)}^{s_{k-1}}) - f(c_{(k)})| < |f(c_{(m-1)}^{s_{m-1}}) - f(c_{(k)})|$ which is precisely (19). Therefore, by Sublemma 2.1 we have

$$(20) \quad |(c_{(k-1)}^{s_{k-1}}, c_{(k)})| \geq Ce^{-\xi(s_k + \dots + s_{m-1})}.$$

By construction we also have $f^{s_0 + \dots + s_{k-1}}(c^j, c_{(0)}) \supset (c_{(k-1)}^{s_{k-1}}, c_{(k)})$ and therefore by the mean value theorem we have

$$(21) \quad |(c^j, c_{(0)})| \geq D^{-(s_0 + \dots + s_{k-1})} |(c_{(k-1)}^{s_{k-1}}, c_{(k)})|.$$

Combining (20) and (21) we get that there exist constants $C, \eta > 0$ such that $d(c^j) = |(c^j, c_{(0)})| \geq Ce^{-\eta(s_0 + \dots + s_{m-1})}$. Finally, from (18) we have that there exists a uniform constant $\nu = 1 + \sum \varepsilon^i$ such that $s_0 + \dots + s_{m-1} \leq \nu s_0 = \nu s(c^j)$, which then gives $d(c^j) = |(c^j, c_{(0)})| \geq Ce^{-\eta\nu s(c^j)}$. This completes the proof in this case.

Step 3. We now consider the situation in which the procedure described in Step 1 leads to the existence of some $0 \leq j \leq q+1$ such that $s_j < \max\{N_0, N_\varepsilon\}$. Since $s_j = s(c_{(j-1)}^{s_{j-1}}, c_{(j)})$ we have that $|c_{(j-1)}^{s_{j-1}} - c_{(j)}| \geq \delta$ where $\delta > 0$ is some constant depending only on f, N_0 and N_ε . Therefore a similar argument to that used in the final part of Step 2 gives the result in this case also. \square

5. TOPOLOGICAL SLOW RECURRENCE IMPLIES CE

We now show that for S-multimodal maps the Topological Slow Recurrence condition implies the Collet-Eckmann condition. We shall be concentrating here on the cylinder sets associated to the *critical values* rather than those associated to the critical points used in the arguments given in the previous section. Consider a critical value c^1 with shadowing times n_i^\pm and corresponding intervals $\hat{I}_\pm^{(n_i^\pm)}$ and $I^{(n_i^\pm)}$ as in section 2.

Definition 3. For any (large) $T > 0$ we say that the gap $n_i^- - n_{i-1}^-$, respectively $n_j^+ - n_{j-1}^+$, between two left, respectively right, shadowing times is *short* if $n_i^- - n_{i-1}^- < T$, respectively $n_j^+ - n_{j-1}^+ < T$. We say that a left short gap and a right short gap are *simultaneous* if they are contained in an interval $\mathcal{T} = [t_1, t_2]$ of times of length $t_2 - t_1 < T$.

In section 5.1 we show that there exist values of T for which there are many intervals of simultaneous short shadowing times. In section 5.2 we show that each such interval implies a uniform estimate on the exponential shrinking of the length of intervals associated to some particular cylinder sets at the corresponding critical value. Finally, in section 5.3 we combine these two results to obtain the Collet-Eckmann property.

5.1. Positive density of simultaneous short shadowing times.

Lemma 3. *For every small $\varepsilon > 0$ there exists $T = T_\varepsilon > 0$, $n_\varepsilon > 0$ and $\eta = \eta_\varepsilon = (1 - 2\varepsilon)/2T_\varepsilon > 0$ such that the number of disjoint simultaneous short (with respect to T) shadowing time intervals associated to any critical values, and occurring before time n , is $\geq \eta n$ for every $n \geq n_\varepsilon$.*

Proof. Suppose that a point x satisfies condition (14). Then for all $\varepsilon > 0$ sufficiently small, there exist $n_\varepsilon, T_\varepsilon$ such that

$$(22) \quad \sum_{\substack{1 \leq n_i^\pm \leq n \\ n_{i+1}^\pm - n_i^\pm > T}} n_{i+1}^\pm - n_i^\pm = \sum_{\substack{1 \leq n_i^\pm \leq n \\ s(x^{n_i^\pm}) > T}} s(x^{n_i^\pm}) \leq \sum_{\substack{1 \leq j \leq n \\ s(x^j) > T}} s(x^j) < \varepsilon n$$

for all $T > T_\varepsilon$ and $n > n_\varepsilon$. Indeed, the equality follows by (13), the first inequality follows simply because the summation on the left is over a subset of times of the summation on the right, and the final inequality follows directly from (14). In particular we have

$$\sum_{\substack{1 \leq n_i^\pm \leq n \\ n_{i+1}^\pm - n_i^\pm \leq T}} n_{i+1}^\pm - n_i^\pm \geq (1 - \varepsilon)n.$$

The calculation holds for both left and right shadowing times independently and therefore there exists at least $(1 - 2\varepsilon)n$ iterates which belong to a small gap for both the left and the right sequence of shadowing times simultaneously. Each such iterate is therefore contained in an interval \mathcal{T} of simultaneous short shadowing times and thus it is possible to define at least $(1 - 2\varepsilon)n/2T_\varepsilon$ disjoint such intervals.

Finally, notice that if some point x satisfies condition (14) then any other point on the orbit of x also satisfies (14). In particular these conclusions hold for all critical values as required. \square

5.2. Exponential shrinking at simultaneous short shadowing times.

Lemma 4. *For any integer $T \geq 1$, there exists a constant $0 < \gamma = \gamma(T) < 1$ such that for any critical value c^1 and any associated pair $[n_{i-1}^-, n_i^-]$ and $[n_{j-1}^+, n_j^+]$ of simultaneous left and right short gaps we have*

$$\frac{|\hat{I}_-^{(n_i^-)}|}{|\hat{I}_-^{(n_{i-1}^-)}|} < \gamma \quad \text{and} \quad \frac{|\hat{I}_+^{(n_j^+)}|}{|\hat{I}_+^{(n_{j-1}^+)}|} < \gamma$$

Proof. Let $\mathcal{T} = [k, \tilde{k}]$ be the interval containing $[n_{i-1}^-, n_i^-]$ and $[n_{j-1}^+, n_j^+]$ and satisfying $\tilde{k} - k < T$. We claim first of all that there exists a $\delta = \delta(T) > 0$ such that

$$|f^k(\hat{I}_-^{(n_{i-1}^-)} \setminus \hat{I}_-^{(n_i^-)})| \geq \delta, \quad |f^k(\hat{I}_+^{(n_{j-1}^+)} \setminus \hat{I}_+^{(n_j^+)})| \geq \delta.$$

To see this, let $\mathcal{C}^{(T)} = \{f^i(\mathcal{C})\}_{i=-T}^T$ denote the set of all images and preimages of the critical set between time $-T$ and $+T$ and define $\delta > 0$ to be the minimum distance between any two points in this set. Now let $[x, y] = \hat{I}_-^{(n_{i-1}^-)} \setminus \hat{I}_-^{(n_i^-)}$ and $[z, w] = \hat{I}_+^{(n_{j-1}^+)} \setminus \hat{I}_+^{(n_j^+)}$. Then, by construction we have

$$\{f^{n_{i-1}^-}(x), f^{n_i^-}(y), f^{n_{j-1}^+}(z), f^{n_j^+}(w)\} \subset \mathcal{C}$$

and therefore, since $n_{i-1}^-, n_i^-, n_{j-1}^+, n_j^+$ are all within T iterates of k we have

$$\{f^k(x), f^k(y), f^k(z), f^k(w)\} \subset \mathcal{C}^{(T)}.$$

These are the endpoints of the intervals $f^k(\hat{I}_-^{(n_{i-1}^-)} \setminus \hat{I}_-^{(n_i^-)})$ and $f^k(\hat{I}_+^{(n_{j-1}^+)} \setminus \hat{I}_+^{(n_j^+)})$ and thus the claim follows. (See Figure 4).

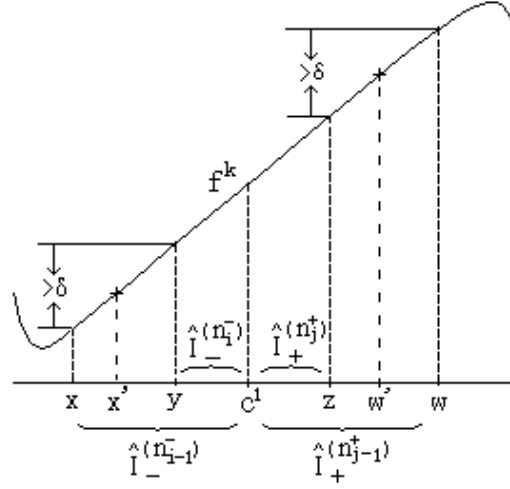


FIGURE 4. Exponential shrinking at simultaneous short shadowing times.

We now choose two points $x' \in (x, y)$ and $w' \in (z, w)$ such that $|f^k[x, x']| = |f^k[x', y]| \geq \delta/2$ and $|f^k[z, w']| = |f^k[w', w]| \geq \delta/2$.

By (11) we observe that the map f^k is monotone on $[x, w]$ and therefore the two extreme intervals $[x, x']$ and $[w', w]$ form the *Koebe space* which guarantees uniformly bounded distortion in $[x', w']$ (see [MelStr93]). In particular the proportion between the lengths of $[x', y]$ and $[y, c^1]$ is uniformly comparable to the proportion between the length of $f^k[x', y]$ and $f^k[y, c^1]$

and therefore we have

$$\begin{aligned} \frac{|\hat{I}_-^{(n_{i-1}^-)}|}{|\hat{I}_-^{(n_i^-)}|} &= \frac{|[x, c^1]|}{|[y, c^1]|} \geq \frac{|[x', c^1]|}{|[y, c^1]|} \\ &\geq 1 + \frac{|[x', y]|}{|[y, c^1]|} \geq 1 + \frac{|f^k[x', y]|}{|f^k[y, c^1]|} \geq 1 + \frac{\Gamma\delta}{2|I|} \end{aligned}$$

for a distortion constant Γ which depends only on δ . This proves the first inequality in the statement of the Lemma with $\gamma = (1 + \frac{\Gamma\delta}{2|I|})^{-1}$. Exactly the same argument gives the second inequality. \square

5.3. Collet-Eckmann. We are now ready to show that all critical values admit exponentially growing derivative along their orbits. More specifically we shall prove the following

Lemma 5. *Suppose that every critical value satisfies (TSR). Then there exist constants $K, \lambda > 0$ such that for every critical value and every $n \geq 1$ we have*

$$r_n := \min \left\{ \frac{|f^n(\hat{I}_-^{(n)})|}{|\hat{I}_-^{(n)}|}, \frac{|f^n(\hat{I}_+^{(n)})|}{|\hat{I}_+^{(n)}|} \right\} \geq Ke^{\lambda n}.$$

The exponential growth of the derivative then follows from the so-called *minimum principle* for maps with negative Schwarzian derivative:

Minimum Principle. *[MelStr93] Let f be an S -multimodal map and $[a, b] \subset I$ be a subinterval. For $i \geq 1$, if $f^i|_{[a,b]}$ is a diffeomorphism, then*

$$|Df^i(x)| \geq \min \left\{ \frac{|f^i(b) - f^i(x)|}{|b - x|}, \frac{|f^i(x) - f^i(a)|}{|x - a|} \right\}$$

for any $x \in (a, b)$.

We therefore get the following

Corollary 5.1. *Suppose that every critical value satisfies (TSR). Then there exist constants $K, \lambda > 0$ such that for every critical value c^1 and every $n \geq 1$ we have*

$$|Df^n(c^1)| \geq Ke^{\lambda n}.$$

We begin with a sublemma which follow easily from the estimates obtained above.

Sublemma 5.1. *For all $n \geq 1$ we have*

$$|\hat{I}_\pm^{(n)}| \leq e^{-\eta n}.$$

Proof. Follows from Lemmas 3 and 4. \square

Proof of Lemma 5. We start by considering a (left or right) shadowing time n_i^\pm . Let $\hat{I}_\pm^{(n_i^\pm)} = [x, c^1]$ and recall that then $f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)}) = (\hat{c}, c^{n_i^\pm+1})$ for some critical point \hat{c} . By lemma 2 we have

$$d(c^j) \geq e^{-s(c^j)/\kappa}.$$

Therefore

$$(23) \quad |f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)})| = |c^{n_i^\pm+1} - \hat{c}| \geq e^{-s(c^{n_i^\pm+1})/\kappa}$$

for every critical value and every shadowing time $n_i^\pm \geq 1$.

We now let n_i^\pm be the smallest (left or right) shadowing time larger than n . Then $n_{i-1}^\pm \leq n < n_i^\pm$ and so $\hat{I}_\pm^{(n_i^\pm)} \subset \hat{I}_\pm^{(n)} = \hat{I}_\pm^{(n_{i-1}^\pm)}$. By (11) we know that $f^{n_i^\pm}|_{\hat{I}_\pm^{(n_{i-1}^\pm)}}$ is monotone, so we have by (23),

$$(24) \quad \begin{aligned} |f^n(\hat{I}_\pm^{(n)})| &\geq D^{-(n_i^\pm-n)} |f^{n_i^\pm}(\hat{I}_\pm^{(n)})| = D^{-(n_i^\pm-n)} |f^{n_i^\pm}(\hat{I}_\pm^{(n_{i-1}^\pm)})| \\ &\geq D^{-(n_i^\pm-n)} |f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)})|. \end{aligned}$$

Then, by (23) and sublemma 5.1, we obtain

$$(25) \quad \begin{aligned} |f^n(\hat{I}_\pm^{(n)})| &\geq D^{-(n_i^\pm-n)} e^{-s(c^{n_i^\pm+1})/\kappa} \geq D^{-(n_i^\pm-n)} e^{-s(c^{n_i^\pm+1})/\kappa} e^{\eta\gamma n} |\hat{I}_\pm^{(n)}| \\ &= D^{-(n_i^\pm-n)} e^{-s(c^{n_i^\pm+1})/\kappa + \eta\gamma n} |\hat{I}_\pm^{(n)}|. \end{aligned}$$

It is therefore sufficient to show that we can choose constants $K, \lambda > 0$ so that

$$r_n \geq D^{-(n_i^\pm-n)} e^{-s(c^{n_i^\pm+1})/\kappa + \eta\gamma n} \geq K e^{\lambda n}.$$

Since $\lim_{n \rightarrow +\infty} s(c^n)/n = 0$, we can choose N sufficiently large such that $s(c^{n_i^\pm+1})/\kappa < \eta\gamma(n_i^\pm + 1)/2$ when $n_i^\pm \geq N$. So

$$r_n \geq D^{-(n_i^\pm-n)} e^{\eta\gamma n - \eta\gamma(n_i^\pm+1)/2}.$$

We start by taking $\tilde{\lambda} = \eta\gamma/2$, fixing some arbitrary $\tilde{\lambda} > \lambda > 0$ and writing $\tilde{\lambda} = \lambda + (\tilde{\lambda} - \lambda)$ and $n_i^\pm = (n_i^\pm - n) + n$. Then we have $e^{\tilde{\lambda}n_i^\pm} = e^{\lambda n} e^{(\tilde{\lambda}-\lambda)n} e^{-\tilde{\lambda}(n-n_i^\pm)}$ and therefore

$$r_n \geq D^{-(n_i^\pm-n)} e^{\eta\gamma n - \eta\gamma(n_i^\pm+1)/2} = D^{-(n_i^\pm-n)} e^{-\tilde{\lambda}(n_i^\pm-n)} e^{-\tilde{\lambda}(\tilde{\lambda}-\lambda)n} e^{\lambda n}.$$

It remains to show that

$$(26) \quad (De^{\tilde{\lambda}})^{-(n_i^\pm-n)} e^{-\tilde{\lambda}(\tilde{\lambda}-\lambda)n} \geq K$$

This follows by the crucial facts that $n_i^\pm - n_{i-1}^\pm > n - n_{i-1}^\pm$ and

$$(27) \quad \lim_{n_{i-1}^\pm \rightarrow +\infty} \frac{s(c^{n_{i-1}^\pm})}{n_{i-1}^\pm} = \lim_{n_{i-1}^\pm \rightarrow +\infty} \frac{n_i^\pm - n_{i-1}^\pm}{n_{i-1}^\pm} = 0$$

by TSR and (13). Indeed, We use (27) which implies that the left side of (26) is bigger than 1 for all $n_i^\pm \geq \tilde{N}$ for some sufficiently large \tilde{N} depending

only on the map and the constants λ and $\tilde{\lambda}$ and thus, again using (27), for all $n \geq N$ for some sufficiently large N also depending only on the same quantities as \tilde{N} . To take care of smaller values of n it is sufficient to choose K sufficiently small depending only on the value of r_n for $n \leq N$. This completes the proof. \square

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DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON
 E-mail address: stefano.luzzatto@imperial.ac.uk
 URL: <http://www.ma.ic.ac.uk/~luzzatto>

ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, CHINESE ACADEMY OF SCIENCES
 E-mail address: lanyu@amss.ac.cn